Journal of Sound and Vibration 330 (2011) 1005–1017

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Natural vibration of circular and annular thin plates by Hamiltonian approach

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ARTICLE INFO

Article history:
Received 7 June 2010
Received in revised form
9 August 2010
Accepted 17 September 2010
Handling Editor: S. Ilanko
Available online 16 October 2010

ABSTRACT

The present paper deals with the natural vibration of thin circular and annular plates using Hamiltonian approach. It is based on the conservation principle of mixed energy and is constructed in a new symplectic space. A set of Hamiltonian dual equations with derivatives with respect to the radial coordinate on one side of the equations and to the angular coordinate on the other side are obtained by using the variational principle of mixed energy. The separation of variables is employed to solve Hamiltonian dual equations of eigenvalue problem. Analytical frequency equations are obtained based on different cases of boundary conditions. The natural frequencies are the roots of the frequency equations and corresponding mode functions are in terms of the dual variables $q_1(r, \theta)$. Three basic edge-constraint cases for circular plates and nine edge-constraint cases for annular plates are calculated and the results are compared well with existing ones.

1. Introduction

In recent years, lightweight plate structures have been widely used in many engineering applications. Components of circular plates and annular plates are commonly used in aeronautical, marine and nuclear structures. In industrial applications, vibration analysis of circular and annular plates is of some practical importance. A vast amount of literature for natural vibration studies of circular plates and annular plates based on two-dimensional theories is available. A comprehensive survey of the previous work is documented in the monograph of Leissa [1]. After that, Narita and Leissa [2,3] analyzed the simply supported plates and free orthotropic elliptical plates by asymptotic expansions and modified Ritz method. Kim and Dickinson [4] obtained an algebraic eigenvalue equation for the free, transverse vibration of thin, annular and sectorial plates based on the Rayleigh–Ritz method with orthogonally generated polynomials as admissible function. Wang and Thevendran [5,6] presented a variant of the Rayleigh–Ritz method for solving the free vibration problem of annular plates with internal axisymmetric supports and extended the method to the problem of elastic buckling of thin annular plates under in-plane radial loads along either free or simply supported with elastic rotational restraints at inner and outer edges. Liew et al. [7] obtained the solutions of free flexural vibration of circular and annular Mindlin plates lying on multiple internal concentric ring supports by the Rayleigh–Ritz method with an admissible displacement function expressed in terms of a set of simple polynomials. Then, Liew et al. [8] developed the method to the buckling and vibration of annular Mindlin plates with internal concentric ring supports subjected to external and internal isotropic in-plane radial pressure. Rajalingham et al. [9] studied the

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0022-460X/$ - see front matter © 2010 Published by Elsevier Ltd.
doi:10.1016/j.jsv.2010.09.015

Nomenclature

D	flexural rigidity
E	Young's modulus
h	thickness
H	Hamiltonian function
L	Lagrangian function
M	equivalent moment
M_r, M_θ	bending moment perpendicular to r-, θ-axis
$M_{r\theta}$	twisting moment
Q	shear force in the θ-direction
Q_r, Q_θ	transverse shear forces
$w(r, \theta)$	natural mode
$W(r, \theta, t)$	deflection
U	potential energy density
K_r, k_θ, $k_{r\theta}$	curvatures
μ_j	eigenvalues of a Hamiltonian operator matrix
$\mu_j^{(\alpha)}$, $\mu_j^{(\beta)}$	two group eigenvalues
υ	Poisson's ratio
ρ	mass density
φ	circumferencial rotation angle
$\boldsymbol{\psi}_j^{(\alpha)}$, $\boldsymbol{\psi}_j^{(\beta)}$	two group eigenfunction-vectors
ω	natural frequency
(x, y)	Cartesian coordinates
(r, θ)	polar coordinate
$\mathbf{H}$	Hamiltonian operator matrix
$\boldsymbol{\Psi}$	state vector in the symplectic space
$\mathbf{q}$, $\mathbf{p}$	mutually dual vectors

plate vibrations by the Rayleigh–Ritz method with the plate characteristic functions as shape functions. Then, Rajalingham et al. [10,11] developed a variational reduction method to analyze the vibration modes and frequency. Chakraverty et al. [12–14] provides natural vibration natural frequencies of plates of various geometries by using two-dimensional boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method. Similarly, Singh and Muhammad [15] used a modified form of the Rayleigh–Ritz method in which the shape functions are used as the admissible displacement fields to solved the natural frequencies and the associated mode shapes. Furthermore, Lim et al. [16] obtained the exact vibration frequencies for transverse shear vibration modes of thick plates by using the state-space technique. It provided a complete spectrum of frequencies, even for very high frequencies without suffering from numerical instability. Wang and Wang [17] analyzed the fundamental frequencies for annular plates with small core sizes. Asymptotic expansion on the exact characteristic equations delineates the singular rise (infinite slope) of the fundamental frequency when the core size is close to zero. By using finite element methods, Bardell et al. [18] describes a general analysis of the vibration characteristics of thin, open, conical isotropic panels using the h-p version of the finite element method in conjunction with Love's thin shell equations. Li and Li [19] analyzed the natural vibration of circular and annular sectorial thin plates using curve strip Fourier p-element. For the smart materials, Ebrahimi and Rastgo [20] investigated the natural vibration behavior of thin circular functionally graded plates integrated with two uniformly distributed actuator layers made of piezoelectric (PZT4) material based on the classical plate theory.

In view of these literatures, it can be seen that all of the methods used the governing equation derived previously in Lagrangian sense involving only one kind of variables in terms of the strain energy. The solution method is based on assumed displacement functions in one or two spatial dimensions. Zhong and his associates [21] developed an analytical symplectic approach for some basic problems in solid mechanics and in elasticity. It is based on the Hamiltonian form with Legendre's transformation. The resulting Hamiltonian dual equations have derivatives with respect to the radial coordinate alone on one side and the angular coordinate alone on the other side. The separation of variables is employed to solve the resulting differential eigenvalue problem and analytical solutions could be obtained by the expansion of eigenfunctions. Unlike the classical semi-inverse methods with pre-assumed trial functions, the symplectic elasticity approach is rigorously rational without any guess functions. All geometric and natural boundary conditions are imposed on the system in a natural manner. It is rational and systematic with a clearly defined, step-by-step derivation procedure. Based on the symplectic approach, many complex problems are solved systematically. Leung et al. [22,23] obtained the analytic stress intensity factors for finite elastic disk and edge-cracked circular piezoelectric shafts using symplectic expansion. After that, Leung et al. [24,25] applied the Hamiltonian approach to solve the thermal stress intensity factors. Lim et al. [26] presented the exact analytical solutions for natural vibration of rectangular thin plates with two opposite edge simply supported. Xing and Liu [27] extend the Lim results [26] for all combinations of boundary conditions.

In the present study the symplectic expansion is used to obtain the natural modes of the Kirchhoff circular and annular plates. With the aid of Hamiltonian principle of mixed energy, a set of Hamiltonian dual equations are obtained. The differential eigenvalue problem is then solved using the separation of variables. The solution is expanded in terms of the symplectic adjoint orthogonal eigensolutions with coefficients to be determined by the boundary conditions. Unlike most research works which are mainly concerned with lower-frequency modes, this method provides low and high mode results for nine basic cases boundary conditions at the inner and outer edges. Numerical comparisons to the classical solutions in literature are presented to validate the efficiency and accuracy of the symplectic method.

2. Basic equations

Consider an isotropic, homogeneous annular plate with uniform thickness h in cylindrical coordinate (r, θ, z) with the z-axis along the longitudinal direction. R_1 and R_2 are the inner radius and outer radius as shown in Fig. 1.

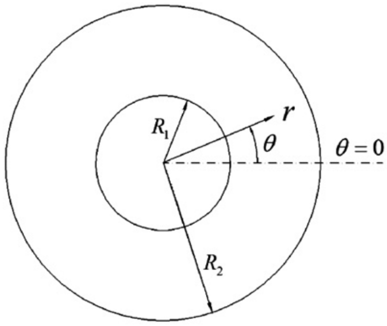

Fig. 1. Geometry and coordinate system of the annular plate.

It is assumed that the thickness h is small compared with in-plane dimensions. We take the (r, θ) plane for the middle plane of the plate and assume the deflection $W(r, \theta, t)$ to be small compared with the thickness h. Define $\partial/\partial r \equiv \partial_r$, $\partial/\partial\theta \equiv \partial_\theta$ and the partial differential equation for free vibrations of thin plates can be written in terms of deflection as [27]

$$\nabla^2\nabla^2 W + \rho h \partial_t^2 W/D = 0 \tag{1}$$

where ρ is the mass density, $D = Eh^3/[12(1-\upsilon^2)]$ is the flexural rigidity, and ∇^2 is the Laplacian. It is well-known that the normal vibrations of an elastic linear system are harmonic, therefore the deflection in normal vibrations of thin plate can be assumed to be

$$W(r,\ \theta,\ t) = w(r,\ \theta)e^{\mathrm{i}\omega t} \tag{2}$$

where ω is the natural frequency and $\mathrm{i}^2 = -1$. Substitution of Eq. (2) into Eq. (1) yields a four order partial differential equation involving natural mode $w(r, \theta)$,

$$\nabla^2\nabla^2 w + \rho h \omega^2 w/D = 0 \tag{3}$$

Based on the Kirchhoff theory, the equations of motion in polar for harmonic natural vibration can be expressed in the frequency domain as

$$\partial_r Q_r + Q_r/r + \partial_\theta Q_\theta/r = -\rho h \omega^2 w \tag{4}$$

where $Q_r = \partial_r M_r + \partial_\theta M_{r\theta}/r + (M_r - M_\theta)/r$; $Q_\theta = \partial_r M_{r\theta} + \partial_\theta M_\theta/r + 2M_{r\theta}/r$. Here M_r, M_θ, $M_{r\theta}$, Q_r and Q_θ are the bending moments, twisting moment and transverse shear forces. The relations of moment resultants and curvatures in the mid-surface of the plate are

$$\begin{cases} M_r = D(\kappa_r + \upsilon\kappa_\theta) \\ M_\theta = D(\kappa_\theta + \upsilon\kappa_r) \\ M_{r\theta} = D(1-\upsilon)\kappa_{r\theta} \end{cases} \tag{5}$$

where υ is Poisson's ratio, E is Young's modulus. The curvatures can be expressed in terms of the displacement as

$$\kappa_r = -\partial_r^2 w, \quad \kappa_\theta = -(r\partial_r w + \partial_\theta^2 w)/r^2, \quad \kappa_{r\theta} = -\partial_r(\partial_\theta w/r) \tag{6}$$

The potential energy density function is presented by

$$U = D[(\kappa_r + \kappa_\theta)^2 - 2(1-\upsilon)(\kappa_r\kappa_\theta - \kappa_{r\theta}^2)]/2 \tag{7}$$

The θ-coordinate in the Lagrangian system is modeled as the time-analogy coordinate in the Hamiltionian system. An over dot is used to indicate differentiation with respect to the θ-coordinate, i.e. $\dot{()} = \partial_\theta$, and further denote the circumferencial rotation angle as

$$\varphi = -\dot{w}/r \tag{8}$$

The Lagrangian function combining the flexural potential energy and the work for external force can be written as

$$\begin{aligned} L(w,\, \dot{w}) &= rU(w,\, \dot{w}) - r\rho h\omega^2 w^2/2 \\ &= Dr\{[\partial_r^2 w + (\partial_r w)/r - \dot{\varphi}/r]^2 - 2(1-\upsilon)[(\partial_r^2 w)((\partial_r w)/r + \ddot{w}/r) - ((\partial_r \dot{w})/r \\ &\quad + \dot{w}/r^2)^2] - (\varphi + \dot{w}/r)[(\partial_r^2 \dot{w})/r + (\partial_r \dot{w})/r^2 + \dot{w}/r^3] - \rho h \omega^2 w^2\}/2 \end{aligned} \tag{9}$$

With the aid of the Hamiltonian principle, for equilibrium

$$\delta \iint_\Omega L(w,\, \dot{w})\,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{10}$$

here Ω is the integral region.

3. Forming the Hamiltonian system

Denote the generalized displacement vector as

$$\mathbf{q}=\{w,\ \varphi\}^{\mathrm{T}}\equiv\{q_1,\ q_2\}^{\mathrm{T}} \tag{11}$$

In the Hamiltonian formulation, the dual (conjugate) vector $\mathbf{p}$ is introduced by the variation of the Lagrangian with respect to $\dot{\mathbf{q}}$

$$\mathbf{p}=\frac{\delta L}{\delta\dot{\mathbf{q}}}=\begin{Bmatrix}-D[(\partial_r^2\dot{w})/r+(\partial_r\dot{w})/r^2+\dot{w}/r^3]\\ -D[\partial_r^2 w+(\partial_r w)/r-\dot{\varphi}/r]\end{Bmatrix}=\begin{Bmatrix}Q\\ M\end{Bmatrix}\equiv\begin{Bmatrix}p_1\\ p_2\end{Bmatrix} \tag{12}$$

Therefore, the dual variables $Q=Q_\theta$ and $M=(M_r+M_\theta)/(1+\upsilon)$ are derived to be the shear force in the θ-direction and equivalent moment respectively. The Hamiltonian function can be obtained by the Legendre transformation

$$H(\mathbf{q},\ \mathbf{p})=\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}}-L(\mathbf{q},\ \dot{\mathbf{q}})=rM^2/(2D)+M\partial_r(r\partial_r w)-r\varphi Q+r\rho h\omega^2 w^2/2 \tag{13}$$

From the Hamiltonian function (13), the fundamental equations, or the dual equations, of the Hamiltonian system are obtained by variation

$$\begin{cases}\dot{\mathbf{q}}=\delta H/\delta\mathbf{p}\\ \dot{\mathbf{p}}=-\delta H/\delta\mathbf{q}\end{cases} \tag{14}$$

or

$$\dot{\boldsymbol{\Psi}}=\mathbf{H}\boldsymbol{\Psi} \tag{15}$$

where $\boldsymbol{\Psi}=\{\mathbf{q},\ \mathbf{p}\}^{\mathrm{T}}$ is the state vector in the symplectic space, and $\mathbf{H}$ is the Hamiltonian matrix operator given by

$$\mathbf{H}=\begin{bmatrix}0 & -r & 0 & 0\\ \partial_r(r\partial_r) & 0 & 0 & r/D\\ \rho h\omega^2 r & 0 & 0 & -\partial_r(r\partial_r)\\ 0 & 0 & r & 0\end{bmatrix} \tag{16}$$

The associated boundary conditions along $r=r_i$ ($i=1,\ 2$) could be one of the following three conditions:

(1). For a clamped edge (C), the deflection and rotation must be zero, i.e.

$$q_1|_{r_i}=0,\quad \partial_r q_1|_{r_i}=0 \tag{17}$$

(2). For a simply supported edge (S), the deflection and bending moment must be zero, i.e.

$$q_1|_{r_i}=0,\quad [\upsilon M-D(1-\upsilon)\partial_r^2 w]|_{r_i}=0 \tag{18}$$

(3). For a free edge (F), the bending moment and total equivalent shear force must be zero, i.e.

$$[\upsilon M-D(1-\upsilon)\partial_r^2 w]|_{r_i}=0,\quad [(2-\upsilon)\partial_r M+(1-\upsilon)M/r+D(1-\upsilon)(r\partial_r w)/r]|_{r_i}=0 \tag{19}$$

For the natural vibration of circular plate, only three kinds of outer edge-constraint conditions are concerned, they are C, S and F. It is more complex for the natural vibration of annular plate since there are nine cases of the edge-constraint conditions, i.e. CC, SS, FF, CS, CF, SC, SF, FC, FS at both the outer and inner edges.

4. Separation of variables and adjoint symplectic orthonormal relations

For the homogeneous Hamiltonian system according to Eq. (15), it is natural to apply the method of separation of variables to reduce it to a differential eigenvalue problem which is very different from the algebraic eigenvalue problem for finding the natural modes of the finite element equations. To this end, the state vector can be expressed as

$$\boldsymbol{\Psi}(r,\ \theta)=\boldsymbol{\psi}_j(r)e^{\mu_j\theta} \tag{20}$$

where μ_j is the unknown eigenvalue and $\boldsymbol{\psi}_j$ is the eigenfunction which has to satisfy the homogenous boundary conditions (17)–(19), so that the differential eigenvalue equation for μ_j and $\boldsymbol{\psi}_j$ is

$$\mathbf{H}\boldsymbol{\psi}_j=\mu_j\boldsymbol{\psi}_j \tag{21}$$

The Hamiltionian operator matrix $\mathbf{H}$ has the following special properties:

If μ_j is an eigenvalue of a Hamiltonian operator matrix, $-\mu_j$ is also an eigenvalue, and there are infinite eigenvalues which can be divided into two sets:

$$(\alpha):\quad \mu_j^{(\alpha)},\ j=1,2,\ldots,\mathrm{Re}(\mu_j^{(\alpha)})>0\ \text{or}\ \mathrm{Re}(\mu_j^{(\alpha)})=0\ \text{and}\ \mathrm{Im}(\mu_j^{(\alpha)})>0 \tag{22}$$

$$(\beta): \quad \mu_j^{(\beta)}, \quad j=1,2,\ldots, \quad \mu_j^{(\beta)}=-\mu_j^{(\alpha)} \tag{23}$$

whose eigenfunction-vectors are denoted, respectively, as $\boldsymbol{\psi}_j^{(\alpha)}$ and $\boldsymbol{\psi}_j^{(\beta)}$. Introduce an inner product $\langle \boldsymbol{\psi}_i, \mathbf{J}, \boldsymbol{\psi}_j \rangle = \int_\Omega (\mathbf{q}_i \mathbf{p}_j - \mathbf{q}_j \mathbf{p}_i)\mathrm{d}\theta$ between any two of them. There are adjoint symplectic orthonormal relations

$$\left\langle \boldsymbol{\psi}_n^{(\alpha)}, \mathbf{J}, \boldsymbol{\psi}_k^{(\alpha)} \right\rangle = \left\langle \boldsymbol{\psi}_n^{(\beta)}, \mathbf{J}, \boldsymbol{\psi}_k^{(\beta)} \right\rangle = 0, \quad \left\langle \boldsymbol{\psi}_n^{(\alpha)}, \mathbf{J}, \boldsymbol{\psi}_k^{(\beta)} \right\rangle = -\left\langle \boldsymbol{\psi}_n^{(\beta)}, \mathbf{J}, \boldsymbol{\psi}_k^{(\alpha)} \right\rangle = \delta_{nk} \tag{24}$$

where $\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}$ is the symplectic identity matrix, δ_{ij} is the Kronecker delta which equals to one when $i=j$ and equals to zero otherwise.

5. Eigensolutions and frequency equations

Homogeneous Eq. (21) with appropriate boundary conditions are discussed in this section. According to the continuity condition at $\theta=0$ and $\theta=2\pi$, it has

$$\boldsymbol{\Psi}(r,0) = \boldsymbol{\Psi}(r,2\pi) \tag{25}$$

Based on Eq. (25), the eigenvalues can be obtained as

$$\mu_n = \mathrm{i}n, \quad \mathrm{i}=\sqrt{-1}, \quad n=0, \pm 1, \pm 2, \ldots \tag{26}$$

5.1. The zero-eigenvalue solutions

Since the solutions of zero-eigenvalues usually have particular physical meaning in the Hamiltonian system, the solutions of zero- and nonzero-eigenvalues should be discussed separately. Consider the equation $\mathbf{H}\boldsymbol{\psi}=\mathbf{0}$, the fundamental solution and the Jordan form solution can be expressed in terms of the zeroth-order Bessel functions and represented by two groups:

$$\boldsymbol{\psi}^{(0,\alpha)} = \begin{Bmatrix} A^0 J_0(\gamma r) + B^0 Y_0(\gamma r) + C^0 I_0(\gamma r) + D^0 K_0(\gamma r) \\ 0 \\ 0 \\ \gamma^4 A^0 J_0(\gamma r) + \gamma^4 B^0 Y_0(\gamma r) + \gamma^4 C^0 I_0(\gamma r) + \gamma^4 D^0 K_0(\gamma r) \end{Bmatrix}$$

$$\boldsymbol{\psi}^{(0,\beta)} = \begin{Bmatrix} [A^0 J_0(\gamma r) + B^0 Y_0(\gamma r) + C^0 I_0(\gamma r) + D^0 K_0(\gamma r)] \\ -[A^0 J_0(\gamma r) + B^0 Y_0(\gamma r) + C^0 I_0(\gamma r) + D^0 K_0(\gamma r)]/r \\ [\gamma^4 A^0 J_0(\gamma r) + \gamma^4 B^0 Y_0(\gamma r) + \gamma^4 C^0 I_0(\gamma r) + \gamma^4 D^0 K_0(\gamma r)]/r \\ [\gamma^4 A^0 J_0(\gamma r) + \gamma^4 B^0 Y_0(\gamma r) + \gamma^4 C^0 I_0(\gamma r) + \gamma^4 D^0 K_0(\gamma r)] \end{Bmatrix} \tag{27}$$

where J_0 and Y_0 are zeroth-order Bessel function of the first kind and second kind, respectively, I_0 and K_0 are zeroth-order modified Bessel function of the first kind and second kind, respectively, $\gamma^4 = \rho h \omega^2 / D$ and A^0, B^0, C^0, D^0 are constants to be undetermined.

Consider the circular plate problem, i.e., $R_1=0$, the displacement at the center of plate are bounded. Since the Bessel functions and modified Bessel function of the second kind are infinite at $r=0$, the solution can be simplified as two groups

$$\boldsymbol{\psi}^{(0,\alpha)} = \begin{Bmatrix} A^0 J_0(\gamma r) + C^0 I_0(\gamma r) \\ 0 \\ 0 \\ A^0 \gamma^4 J_0(\gamma r) + C^0 \gamma^4 I_0(\gamma r) \end{Bmatrix}, \quad \boldsymbol{\psi}^{(0,\beta)} = \begin{Bmatrix} [A^0 J_0(\gamma r) + C^0 I_0(\gamma r)]\theta \\ -[A^0 J_0(\gamma r) + C^0 I_0(\gamma r)]/r \\ [A^0 \gamma^4 J_0(\gamma r) + C^0 \gamma^4 I_0(\gamma r)]/r \\ [A^0 \gamma^4 J_0(\gamma r) + C^0 \gamma^4 I_0(\gamma r)]\theta \end{Bmatrix} \tag{28}$$

It should be pointed out that the Jordan form solutions do not satisfy the continuity condition (25), so it should be eliminated in the following parts. The physical interpretations of the fundamental solution (27) and (28) are the axis-symmetric natural vibration solutions for circular annular and circular plate problems, respectively.

5.2. The nonzero-eigenvalue solutions

In this section, the nonzero-eigenvalues solutions will be studied. Substituting $\mu_n=\mathrm{i}n$ $(n \neq 0)$ into Eq. (21), the eigensolutions can be expressed in terms of the Bessel functions

$$\boldsymbol{\psi}_n = \begin{Bmatrix} A_1 J_n(\gamma r) + B_1 Y_n(\gamma r) + C_1 I_n(\gamma r) + D_1 K_n(\gamma r) \\ A_2 J_n(\gamma r)/r + B_2 Y_n(\gamma r)/r + C_2 I_n(\gamma r)/r + D_2 K_n(\gamma r)/r \\ A_3 J_n(\gamma r)/r + B_3 Y_n(\gamma r)/r + C_3 I_n(\gamma r)/r + D_3 K_n(\gamma r)/r \\ A_4 J_n(\gamma r) + B_4 Y_n(\gamma r) + C_4 I_n(\gamma r) + D_4 K_n(\gamma r) \end{Bmatrix} \tag{29}$$

where J_n and Y_n are Bessel function of the first kind and second kind, respectively, I_n and K_n are modified Bessel function of the first kind and second kind, respectively, $\gamma^4=\rho h\omega^2/D$ and A_i, B_i, C_i, D_i (i=1,2,3,4) are constants to be undetermined by the boundary conditions (17)–(19). Substituting the solutions (29) into Eq. (21), the following relationships among the unknown constants in terms of $\{A_1, B_1, C_1, D_1\}$ are found:

$$\begin{gathered}A_2=-\mathrm{i}n/A_1,\quad B_2=-\mathrm{i}n/B_1,\quad C_2=-\mathrm{i}n/C_1,\quad D_2=-\mathrm{i}n/D_1,\quad A_3=\mathrm{i}n\gamma^4A_1,\quad B_3=\mathrm{i}n\gamma^4B_1,\\ C_3=-\mathrm{i}n\gamma^4C_1,\quad D_3=-\mathrm{i}n\gamma^4D_1,\quad A_4=\gamma^4A_1,\quad B_4=\gamma^4B_1,\quad C_4=\gamma^4C_1,\quad D_4=\gamma^4D_1\end{gathered}\tag{30}$$

It should be pointed out that, the eigensolution (29) for the natural vibration of annular plate can be reduced to that of circular plate. Consider, if R_1=0, that the displacement at the center of plate are bounded. However, the Bessel functions and modified Bessel function of the second kind are infinite at r=0, thus B_j=D_j=0 (j=1, 2, 3, 4). The eigensolution for the circular plate are represented in the form:

$$\boldsymbol{\psi}_n=\begin{Bmatrix}A_1J_n(\gamma r)+C_1I_n(\gamma r)\\ A_2J_n(\gamma r)/r+C_2I_n(\gamma r)/r\\ A_3J_n(\gamma r)/r+C_3I_n(\gamma r)/r\\ A_4J_n(\gamma r)+C_4I_n(\gamma r)\end{Bmatrix}\tag{31}$$

where the constants A_j and C_j (j=1, 2, 3, 4) are related by Eq. (30). The physical interpretations of Eqs. (29) and (31) are the asymmetric natural vibration solutions for circular annular and circular plate problems, respectively.

For convenience, the solutions (27), (28), (29) and (31) can be expressed in a unified form:

$$\boldsymbol{\psi}_n=\begin{Bmatrix}A_1J_n(\gamma r)+\Delta(R_1)B_1Y_n(\gamma r)+C_1I_n(\gamma r)+\Delta(R_1)D_1K_n(\gamma r)\\ A_2J_n(\gamma r)/r+\Delta(R_1)B_2Y_n(\gamma r)/r+C_2I_n(\gamma r)/r+\Delta(R_1)D_2K_n(\gamma r)/r\\ A_3J_n(\gamma r)/r+\Delta(R_1)B_3Y_n(\gamma r)/r+C_3I_n(\gamma r)/r+\Delta(R_1)D_3K_n(\gamma r)/r\\ A_4J_n(\gamma r)+\Delta(R_1)B_4Y_n(\gamma r)+C_4I_n(\gamma r)+\Delta(R_1)D_4K_n(\gamma r)\end{Bmatrix}\tag{32}$$

in which $\Delta(R_1)=1-\delta(R_1)$, $\delta(R_1)$ is the Dirac Delta function which equals to one when R_1=0 and equals to zero otherwise. After the eigensolutions have been obtained in the form of Eq. (32), the remaining problems are to determine the four unknown constants A_1–D_1 and the frequency equation by means of the boundary conditions. Substituting the solutions (32) into the boundary conditions (17)–(19), one has

$$\mathbf{E}\boldsymbol{\chi}=0\tag{33}$$

where $\boldsymbol{\chi}=\{A_1, B_1, C_1, D_1\}^{\mathrm{T}}$, $\mathbf{E}$ is the matrix to be established by the boundary conditions along r=R_1 and R_2. For obtaining nontrivial solutions, the determinant of coefficients matrix of Eq. (33) must be zero, leading to the following equation:

$$|\mathbf{E}|=0\tag{34}$$

Eq. (34) is the frequency equation and the frequencies of the natural vibration are the roots of the transcendental equation. There are infinite numbers of modes for each set of eigensolution n. When denoting m as the number of modes taken for eigensolution n, there will be mn total number of terms. After solving the frequency equation (34), the nontrivial solution of Eq. (33) can be represented by linear combinations of the modes and the remaining unknown constants $\{A_1, B_1, C_1, D_1\}$ are determined up to a normalization constant. The normal mode function $w(r,\theta)$ will be determined also.

To illustrate the application of the symplectic methodology for natural vibration problems, for example, a circular plate here is assumed clamped at the outer edge. According the boundary condition (17), Eq. (33) becomes

$$\begin{bmatrix}J_n(\gamma R_2) & I_n(\gamma R_2)\\ n/R_2J_n(\gamma R_2)-\gamma J_{n+1}(\gamma R_2) & n/R_2I_n(\gamma R_2)-\gamma I_{n+1}(\gamma R_2)\end{bmatrix}\begin{Bmatrix}A_1\\ C_1\end{Bmatrix}=0\tag{35}$$

The corresponding frequency equation is obtained from Eq. (34),

$$I_n(\gamma R_2)J_{n+1}(\gamma R_2)+J_n(\gamma R_2)I_{n+1}(\gamma R_2)=0\tag{36}$$

The roots γ of Eq. (36) are the frequency of natural vibration for the completely clamped circular plate. From Eq. (35), we have

$$C_1=A_1J_n(\gamma R_2)/I_n(\gamma R_2)\tag{37}$$

Letting A_1=1, we can determine the normal mode function for each frequency γ

$$w(r,\theta)=[J_n(\gamma r)+J_n(\gamma R_2)I_n(\gamma r)/I_n(\gamma R_2)]e^{\mathrm{i}n\theta}\tag{38}$$

For other boundary conditions including the other cases for the circular plates and the nine cases for the annular plates, similar results will be obtained in the same way.

Finally, the solutions of the homogenous equation (21) can be the linear combinations of eigensolutions,

$$\boldsymbol{\Psi}=\sum_{m=1}\left(a_m^{(0)}\boldsymbol{\psi}_m^{(0,\,\alpha)}+b_m^{(0)}\boldsymbol{\psi}_m^{(0,\,\beta)}\right)+\sum_{m=1}\left(\sum_{n=1}a_{mn}\boldsymbol{\psi}_{mn}^{(\alpha)}e^{\mu_n^{(\alpha)}\theta}+\sum_{n=0}b_{mn}\boldsymbol{\psi}_{mn}^{(\beta)}e^{\mu_n^{(\beta)}\theta}\right)\tag{39}$$

with undetermined coefficients $a_m^{(0)}$, $b_m^{(0)}$, a_{mn} and b_{mn} to be determined by the initial conditions.

Table 1
Comparison of the present results with classical plate theory for the lowest six natural frequency parameters $\omega R_2\sqrt{\rho h/D}$ (υ=0.3).

Results	Mode number					
	1	2	3	4	5	6
Circular plate with free boundary condition						
Sato [28]	5.3592	9.0120	–	–	–	–
Narita [3]	5.3583	9.0031	12.439	20.475	–	–
Leissa [1]	5.253	9.084	12.23	20.52	–	–
Present	5.3583	9.0031	12.439	20.475	21.835	33.495
Circular plate with hard simply supported boundary condition						
Leissa and Narita [2]	4.9352	13.898	25.613	29.720	39.957	48.479
Kim and Dickinson [4]	4.9352	13.898	25.613	29.720	39.958	48.480
Li and Li [19]	–	13.898	25.613	–	39.957	48.479
Present	4.9352	13.898	25.615	29.719	39.957	48.479
Circular plate with clamped boundary condition						
Carrington [29]	10.216	21.260	34.880	39.771	51.040	60.820
Kim and Dickinson [4]	10.216	21.260	34.877	39.771	51.030	60.829
Leissa [1]	10.216	21.26	34.88	39.771	51.04	60.82
Present	10.216	21.260	34.877	39.771	51.031	60.829

Table 2
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the circular plate with free boundary condition (υ=0.3).

n	Mode number				
	1	2	3	4	5
0	9.0031	38.4432	87.7502	156.8183	245.6335
1	20.4745	59.8116	118.9573	197.8718	296.5401
2	5.3583	35.2601	84.3661	153.3059	242.0361
3	12.4390	53.0078	111.9450	190.6918	289.2378
4	21.8352	73.5426	142.4309	231.0305	339.4128
5	33.4949	96.7553	175.7353	274.2522	392.5053

Table 3
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the circular plate with hard simply supported boundary condition (υ=0.3).

n	Mode number				
	1	2	3	4	5
0	4.9352	29.7193	74.1561	138.3203	222.2150
1	13.8983	48.4786	102.7738	176.8012	270.5676
2	25.6148	70.1169	134.2978	218.2023	321.8417
3	39.9574	94.5491	168.6749	262.4847	376.0121
4	56.8422	121.7027	205.8525	309.6073	433.0487
5	76.2034	151.5182	245.7780	359.5353	492.9189

Table 4
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the circular plate with clamped boundary condition (υ=0.3).

n	Mode number				
	1	2	3	4	5
0	10.2158	39.7711	89.1045	158.1841	247.0064
1	21.2604	60.8287	120.0807	199.0534	297.7601
2	34.8770	84.5826	153.8150	242.7205	351.3370
3	51.0306	111.0218	190.3037	289.1799	407.7295
4	69.6666	140.1084	229.5200	338.4112	466.9250
5	90.7391	171.8030	271.4280	390.3924	528.9021

6. Numerical results and comparisons

Many results have been reported. However, most of these works are mainly concerned with lower-mode frequency parameters or the fundamental frequency for certain boundary conditions. The method of the present paper can deal with all kinds of combinations of free, clamped and simply supported boundary conditions. The three edge-constraint cases for circular plates and the nine edge-constraint cases for annular plates are analyzed in this section. Natural frequency parameters are listed in tables. The accuracy of these results is verified by existing solutions.

6.1. Natural frequencies for the circular plate

Table 1 tabulates the comparison of the lowest six frequency parameters for circular plates with υ=0.3. It is seen that the present results for all the three cases boundary conditions are in excellent agreement with the classical results. The rigid body modes are not shown in the tables. It is interesting to note the two natural modes that has been missed by Li and Li [19] for the circular plate with hard simply supported boundary condition.

Tables 2–4 tabulate the first five non-dimensional frequencies for each eigensolution n in Eq. (32). In general, it is seen in Tables 3 and 4 that the natural frequencies are monotonically increasing with increasing eigensolution number n. It is not so for the free boundary condition in Table 2. It is shown that very high modes can be calculated without difficulties.

Table 5

Comparison of the present results with classical plate theory for the lowest six natural frequency parameters $\omega R_2\sqrt{\rho h/D}$ (υ=1/3, R_1/R_2=0.4, ANSYS: h/R_2=0.001).

Results	Mode number					
	1	2	3	4	5	6
Annular plate with clamped outer and inner edge (CC)						
Chakraverty et al. [14]	61.88	63.04	66.87	74.97	86.88	105.7
ANSYS	61.700	62.809	66.433	73.291	84.096	99.195
Present	61.872	62.996	66.672	73.630	84.594	99.904
Annular plate with simply supported outer and inner edge (SS)						
Chakraverty et al. [14]	28.08	30.09	36.27	47.23	61.69	80.11
ANSYS	28.127	30.122	36.166	46.263	60.211	77.654
Present	28.184	30.079	36.143	46.321	60.423	78.089
Annular plate with free outer and inner edge (FF)						
Chakraverty et al. [14]	4.533	8.583	11.77	17.80	21.27	32.92
ANSYS	4.6068	8.5837	11.945	17.248	21.562	31.591
Present	4.5325	8.5510	11.765	17.043	21.262	31.356
Annular plate with clamped outer and simply supported inner edge (CS)						
Chakraverty et al. [14]	44.93	46.74	52.39	62.60	77.23	96.32
ANSYS	44.982	46.790	52.396	62.123	76.047	93.928
Present	44.932	46.735	52.353	62.148	76.230	94.342
Annular plate with clamped outer and free inner edge (CF)						
Chakraverty et al. [14]	13.50	19.46	31.74	47.81	66.81	72.00
ANSYS	13.566	19.609	31.588	46.970	65.904	66.448
Present	13.500	19.389	31.338	46.855	65.984	66.924
Annular plate with simply supported outer and clamped inner edge (SC)						
Chakraverty et al. [14]	41.27	42.63	47.52	56.04	68.66	87.43
ANSYS	41.155	42.428	46.529	54.063	65.496	80.901
Present	41.261	42.548	46.699	54.332	65.918	81.530
Annular plate with simply supported outer and free inner edge (SF)						
Chakraverty et al. [14]	4.748	12.06	23.56	37.91	47.32	53.27
ANSYS	4.7598	12.037	23.178	37.222	47.115	52.890
Present	4.7436	11.907	23.098	37.272	47.282	52.890
Annular plate with free outer and clamped inner edge (FC)						
Chakraverty et al. [14]	9.082	9.176	10.53	15.34	23.33	34.67
ANSYS	9.0151	9.1102	10.450	14.925	22.918	33.917
Present	9.0719	9.1294	10.366	14.726	22.530	33.455
Annular plate with free outer and simply supported inner edge (FS)						
Chakraverty et al. [14]	3.634	4.001	6.918	13.26	22.06	33.45
ANSYS	3.6782	4.0221	6.8359	13.040	22.054	33.549
Present	3.6671	3.9221	6.6546	12.793	21.464	33.054

6.2. Natural frequencies for the annular plate

Table 5 presents the comparison of the lowest six frequency parameters for the annular plates with $\upsilon=1/3$. It is observed that the present results for the nine cases boundary conditions are almost identical to those reported by Chakraverty et al. [14] and the ANSYS results. The ANSYS results are calculated by shell 63 elements having $h/R_2=0.001$. However, the errors of the natural frequencies predicted by Chakraverty et al. [14] are increasing monotonically with the mode number in all

Table 6
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with clamped outer and inner edge (CC) ($\upsilon=1/3$).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	34.609	61.872	139.61	559.16
1	36.103	62.996	140.48	559.84
2	41.820	66.672	143.13	561.89
3	53.388	73.630	147.79	565.33
4	70.501	84.594	154.83	570.24
5	90.991	99.904	164.31	576.68

Table 7
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with simply supported outer and inner edge (SS) ($\upsilon=1/3$).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	16.733	28.183	62.122	247.07
1	19.432	30.079	63.681	248.28
2	27.285	36.143	68.395	252.00
3	40.340	46.321	76.103	258.17
4	57.120	60.423	87.160	266.85
5	76.261	78.087	101.16	278.26

Table 8
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with free outer and inner edge (FF) ($\upsilon=1/3$).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	8.4419	8.5510	10.548	18.262
1	19.675	17.043	18.192	29.394
2	5.0508	4.5325	3.8641	3.1970
3	12.187	11.765	10.561	8.8719
4	21.209	21.262	20.233	16.901
5	32.967	32.862	32.242	27.288

Table 9
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with clamped outer and simply supported inner edge (CS) ($\upsilon=1/3$).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	26.619	44.932	98.793	389.49
1	29.158	46.735	100.08	390.46
2	37.579	52.353	104.02	393.37
3	51.685	62.148	110.72	398.24
4	69.807	76.230	120.40	405.11
5	90.768	94.342	133.41	414.08

cases. Moreover, the natural frequency parameters of the annular plates with different inner-to-outer radius ratio R_1/R_2=0.2(0.2)0.8, m=1 and n=0(1)5 are calculated and presented in Tables 6–14. It is found the fundamental natural frequency increases with increasing radius ratio R_1/R_2 in most kinds of boundary conditions when n=0 except the free–free boundary conditions that the natural frequency parameter decreases monotonically from 5.0508 to 3.1970 when the radius ratio R_1/R_2 increases from 0.2 to 0.8. The frequency parameter increases monotonically with respect to n for most cases except the free–free boundary conditions where the frequency parameter changes quite irregularly. For higher modes, similar phenomena can be observed, that is, the frequency parameter increases monotonically with the radius ratio R_1/R_2, except for the free–free and clamp–free boundary conditions where changes can be quite irregular.

Table 10
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with clamped outer and free inner edge (CF) (υ=1/3).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	10.347	13.500	25.541	92.816
1	20.492	19.389	28.651	94.248
2	33.764	31.338	36.420	98.780
3	50.531	46.855	48.040	106.17
4	69.549	65.984	62.941	116.16
5	90.714	88.141	81.147	128.98

Table 11
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with simply supported outer and clamped inner edge (SC) (υ=1/3).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	22.767	41.261	94.264	381.64
1	24.322	42.548	95.296	382.52
2	30.088	46.699	98.564	385.18
3	41.329	54.332	104.19	389.62
4	57.405	65.918	112.73	395.95
5	76.308	81.530	123.47	404.17

Table 12
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with simply supported outer and free inner edge (SF) (υ=1/3).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	4.7325	4.7436	5.6630	9.4549
1	13.583	11.907	11.728	16.787
2	24.935	23.098	21.962	29.654
3	39.710	37.272	34.758	44.111
4	56.969	54.683	49.936	59.826
5	76.243	74.954	68.043	77.125

Table 13
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with free outer and clamped inner edge (FC) (υ=1/3).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	5.2135	9.0719	20.607	84.698
1	4.8171	9.1294	20.976	85.321
2	6.3431	10.366	22.275	87.265
3	12.395	14.726	25.716	90.470
4	21.233	22.530	32.533	95.343
5	32.969	33.456	40.562	102.12

Table 14
Dimensionless frequency parameters $\omega R_2\sqrt{\rho h/D}$ for the annular plate with free outer and simply supported inner edge (FS) (υ=1/3).

n	R_1/R_2			
	0.2	0.4	0.6	0.8
0	3.3133	3.6671	4.8087	8.7818
1	2.8425	3.9221	6.0442	12.482
2	5.5344	6.6546	9.7163	20.151
3	12.251	12.793	15.792	29.513
4	21.211	21.464	24.256	40.422
5	32.967	33.054	35.094	52.581

Table 15
Comparison of the present results to 3D solutions for the lowest six natural frequency parameters $\omega R_2\sqrt{\rho h/D}$ of the circular plate (υ=0.3 [30,31]: h/R_2=0.01).

Results	Mode number					
	1	2	3	4	5	6
Circular plate with free boundary condition						
Liew and Yang [30]	5.3570	9.0018	12.433	20.466	21.820	33.463
Han and Liew [32]	5.2634	9.0766	12.240	20.521	-	-
Present	5.3583	9.0031	12.439	20.475	21.835	33.495
Circular plate with simply supported boundary condition						
Liew and Yang [30]	4.9360	13.894	25.603	29.706	39.930	48.439
Present	4.9352	13.898	25.615	29.719	39.957	48.479
Circular plate with clamped boundary condition						
Liew and Yang [30]	10.250	21.326	34.974	39.878	51.155	60.968
Han and Liew [32]	10.226	21.276	34.892	39.785	51.033	60.823
Present	10.216	21.260	34.877	39.771	51.031	60.829

Table 16
Comparison of the present results to 3D solutions for the axisymmetric frequency parameters $\omega R_2\sqrt{\rho h/D}$ of the annular plate (υ=0.3 [32], h/R_2=0.001, R_1/R_2=0.4).

Edges	Results	Mode number				
		1	2	3	4	5
CC	Han and Liew [32]	61.871	170.89	335.34	554.59	828.64
	Present	61.872	170.90	335.37	554.66	828.80
SC	Han and Liew [32]	41.193	137.15	287.88	493.44	753.80
	Present	41.261	137.11	287.96	493.61	753.92
FC	Han and Liew [32]	9.0205	58.548	168.68	333.05	552.29
	Present	9.0719	58.672	168.80	333.18	552.36
CS	Han and Liew [32]	45.044	140.93	291.74	497.35	757.74
	Present	44.932	140.81	291.64	497.45	757.86
SS	Han and Liew [32]	28.122	110.56	247.69	439.61	686.32
	Present	28.184	110.50	247.51	439.58	686.41
FS	Han and Liew [32]	3.6727	42.556	138.68	289.46	495.06
	Present	3.6671	42.556	138.69	289.48	495.10
CF	Han and Liew [32]	13.603	67.157	177.01	341.36	560.58
	Present	13.500	66.925	176.77	341.13	560.65
SF	Han and Liew [32]	4.7640	47.464	143.25	293.93	499.45
	Present	4.7436	47.282	143.06	293.76	499.49
FF	Han and Liew [32]	8.6139	65.681	174.71	339.09	558.30
	Present	8.5510	65.567	174.58	338.96	552.36

6.3. Comparison to the 3D results

For further validations, the present results are compared to that obtained based on the three-dimensional (3D) elasticity solutions. Table 15 presents the lowest six natural frequency parameters of circular plates for all the three case boundary conditions with an aspect of h/R_2=0.01. It is seen that the present results using the proposed symplectic method agree well with 3D results calculated by Liew and Yang [30] and Zhou et al. [31]. In addition, the axisymmetric frequency parameters (n=0) for the annular plates are compared with the 3D frequency solutions obtained by Han and Liew [32] in Table 16. Excellent agreement is also recorded in Table 16. All the comparison studies above proved a solid demonstrations that the present symplectic elasticity approach is very efficient for exact analysis of circular and annular plates natural vibration base on the classical Kirchhoff plate theory.

7. Conclusion

In this paper, a new Hamiltonian approach has been employed to solve natural vibration of circular and annular thin plates. The Hamiltonian dual equations were derived based on the mixed energy Hamiltonian function. The Hamiltonian equation was then translated into a differential eigenvalue equation in the in-plane domain of the plate, and it was then analytically solved with rigorous derivation using the method of variable separation and expansion of eigensolutions. The zero- and nonzero-eigenvalues solutions represented the symmetrical and asymmetrical natural vibration, respectively. The exact normal modes and frequency equations can be obtained for any combination of boundary conditions. The natural frequencies are calculated for these plates with all possible combinations of free, clamped and simply supported boundary conditions and inner-to-outer radius ratios. These results can be used to validate the accuracy of other numerical method as benchmark values. The application of the symplectic dual method are not limited to free vibration problems of the circular and annular plates, and can be extended to forced vibration problems, the static and dynamic problems of thick plate and others.

Acknowledgment

The support of the National Basic Research Program of China (973 Program, Grant no. 2009CB724302) is gratefully acknowledged.

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